Differential Equation

Ordinary Differential Equation of Higher order

Module - I

This notes will cover your semester – II syllabus

Booklet - II

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Syllabus

1. Linear differential equation of n^{th} order with constant coefficients
2. Simultaneous linear differential equation
3. Second order linear differential equations with variable coefficients
4. Solution by changing independent variable
5. Reduction of order
6. Normal form
7. Method of variation of parameter
8. Cauchy – Euler equation
9. Series solutions (Frobenius Method)

Linear Differential Equations of Second and Higher order

A differential equation of the form $\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + ...+ P_{n-1}\frac{dy}{dx} + P_n = Q(x)$ ……(1) is called as a linear differential equation of order n with constants coefficients. Where $P_1, P_2 ....P_n$ are real constants.

- $\frac{d}{dx} \equiv D, \frac{d^2}{dx^2} \equiv D^2, ... \frac{d^n}{dx^n} \equiv D^n$
- Equation (1) is also written as $f(D)y = Q(x), where\ f(D) = D^n + P_1 D^{n-1} + ...+ P_{n-1}D + P_n$
- The General Solution of the above equation is y = C.F .+ P.I. or **$y = y_c + y_p$**
  C.F. = complementary function & P.I. = Particular Integral or function
- **Auxillary Equation(A.E.)**
  An equation of the form $f(m) = 0$ (this we will get by replacing D by m in $f(D)$) is a polynomial equation, by solving this we get roots $m_1, m_2, ...m_n$.

Complementary Function

The general solution of $f(D)y = 0$ is called as Complementary function and is denoted by y_c and it depends upon the nature of roots of $f(m) = 0$.

- If the roots are real and distinct say $m_1, m_2, ...m_n$
- Then $y_c = c_1 e^{m_1 x} + ... + c_n e^{m_n x}$
- If two or more roots are equal say $m_1 = m_2 = ... = m_n \Rightarrow y_c = e^{m_1 x}\left(c_1 + c_2 x + ... + c_n x^{n-1}\right)$
- If roots of A.E. are complex i.e. $\alpha \pm \beta$ then $y_c = e^{\alpha x}\left(c_1 \cos \beta x + c_2 \sin \beta x\right)$ for repeated complex roots same procedure will be applied as of equal roots.
- If roots of A.E. are in the form of surds i.e. $m = \alpha \pm \sqrt{\beta}$, $y_c = e^{\alpha x}\left(c_1 \cosh \sqrt{\beta} x + c_2 \sinh \sqrt{\beta} x\right)$
- If repeated roots of surds say
  $m = \alpha \pm \sqrt{\beta}, \alpha \pm \sqrt{\beta} \quad \Rightarrow y_c = e^{\alpha x}\left[\left(c_1 + c_2 x\right)\cosh \sqrt{\beta} x + \left(c_3 + c_4\right)\sinh \sqrt{\beta} x\right]$

Particular Integral

The evaluation of $\frac{1}{f(D)}Q(x)$ is called as Particular Integral and it is denoted by $\boldsymbol{y_p}$ i.e. $y_p = \frac{1}{f(D)}Q(x)$

Methods to find Particular Integral

Method 1: P.I. of $f(D)y = Q(x)$ where $Q(x) = e^{ax}$ where a is constant

In this case

$$y_p = \frac{1}{f(D)}e^{ax} \Rightarrow y_p = \frac{1}{f(a)}e^{ax} if\ f(a) \neq 0$$

$$if\ \ f(a) = 0 \quad then\ y_p = e^{ax}\frac{1}{f(D+a)}$$

Method 2: P.I. of $f(D)y = Q(x)$ where $Q(x) = \sin ax\, or \cos ax, a$ is constant

Step - I replace $D^2\ by - a^2\ in\ f(D)\, if\ \ f(-a^2) \neq 0$ then $y_p = \frac{1}{f(-a^2)}\sin ax\ or\ \cos ax$

If $f(-a^2) = 0\ then\ y_p = \frac{x}{2}\int \sin ax\, dx\ or\ \frac{x}{2}\int \cos ax\, dx\ respectively$

Method 3: P.I. of $f(D)y = Q(x)$ where $Q(x) = x^k, k \in Z^+$ is constant

In this case $y_p = [f(D)]^{-1}x^k$ expand $[f(D)]^{-1}$ by the binomial theorem in ascending powers D as for as operation on x^k is zero.

Expanding this relation upto kth derivative by using Binomial expansion and hence get y_p .

Important Formulae:

1. $(1-D)^{-1} = 1 + D + D^2 + ...$
2. $(1+D)^{-1} = 1 - D + D^2 - .....$
3. $(1-D)^{-n} = 1 + nD + \frac{n(n-1)}{2!}D^2 + ...$
4. $(1+D)^{-n} = 1 - nD + \frac{n(n-1)}{2!}D^2 - .....$

Method 4: P.I. of $f(D)y = Q(x)$ where $Q(x) = e^{ax} V$ *where V is function of x and a* is constant

In this case

$$y_p = \frac{1}{f(D)} e^{ax} V$$

$$y_p = e^{ax} \frac{1}{f(D+a)} V$$

Now we will proceed further according to nature of V.

Method 5: P.I. of $f(D)y = Q(x)$ where

$Q(x) = x^k$ *where v is any function of x* $(i.e. \sin ax$ *or* $\cos ax)$ *where* $k \in Z^+$ *and a* **is constant**

We know that $y_p = \frac{1}{f(D)} x^k v$, Case – I Let $k = 1$, *then* $y_p = \left[x - \frac{f'(D)}{f(D)} \right] \frac{1}{f(D)} v$

Now for the Case – II $k \neq 1 \& v = \sin ax$ *or* $\cos ax$ we will use $e^{i\theta} = \cos\theta + i\sin\theta$ where R.$P.\cos\theta \& I.P.\sin\theta$

$y_p = \frac{1}{f(D)} x^k \sin ax$ $y_p = \frac{1}{f(D)} x^k I.P.\left(e^{iax}\right)$ $y_p = I.P.e^{iax} \frac{1}{f(D+ia)} x^k$ By using previous methods we will solve further.	$y_p = \frac{1}{f(D)} x^k \cos ax$ $y_p = \frac{1}{f(D)} x^k R.P.\left(e^{iax}\right)$ $y_p = R.P.e^{iax} \frac{1}{f(D+ia)} x^k$ By using previous methods we will solve further.

General Method: P.I. of $f(D)y = Q(x)$ where Q(x) is function of x.

In this case if f (D) =(D –a) then $y_p = \frac{1}{D-a}Q(x) = e^{ax}\int e^{-ax}Q(x)dx$

Similarly f(D) = D + a then $y_p = \frac{1}{D+a}Q(x) = e^{-ax}\int e^{ax}Q(x)dx$

This method is used for the problems of the following type

- $\left(D^2 - 3D + 2\right)y = \sin\left(e^{-x}\right)$
- $\left(D^2 + a^2\right)y = \sec ax$
- $\left(D^2 + a^2\right)y = \tan ax$
- $\left(D^2 + a^2\right)y = co\sec ax$

Cauchy's Linear Equations (or) Homogenous Linear Equations

A differential equation of the form $\left[x^n D^n + A_1 x^{n-1} D^{n-1} + ... + A_{n-1} xD + A_n\right] y = Q(x)$ is called n[th] order Cauchy's Linear Equation in terms of dependent variable y and independent variable x, where

$A_1, A_2, ...A_n$ are real constants.

Substitute $x = e^z \Rightarrow \log x = z$ and $xD = \theta, x^2D^2 = \theta(\theta - 1)......\&\, \theta = \frac{d}{dz}$. then above relation becomes $f(\theta)y = Q(z)$ which is linear D.E. with constant coefficients, by using previous methods, we can find complementary function and particular integral of it, and hence by replacing z = log x we get the required General Solution of Cauchy's Linear Equation.

Legendre's Linear equation

An differential of the form $\left[(ax+b)^n D^n + A_1 (ax+b)^{n-1} D^{n-1} + ... + A_{n-1}(ax+b)D + A_n\right] y = Q(x)$ is called Legendre's linear equation of order n, where $a, b, A_1, A_2, ...A_n$ are real constants.

Now substitute $(ax+b) = e^z \Rightarrow z = \log(ax+b)$ and

$(ax+b)D = a\,\theta, (ax+b)^2 D^2 = a^2\theta(\theta-1)......\&\, \theta = \frac{d}{dz}$, then above relation becomes $f(\theta)y = Q(z)$ which is linear D.E. with constant coefficients, by using previous methods, we can find complementary function and particular integral of it, and hence by replacing z = log (ax + b) we get the required General Solution of Legendre's Linear Equation.

Method of Variation of Parameters

To find the general solution of $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Q\,y = R(x)$.......(1)

Let the complementary function of the above equation is $y_c = c_1 u + c_2 v$

Let Particular Integral $y_p = Au + Bv$, where

$$A = \int \frac{-v\,R}{uv' - u'v}\,dx \;\;\&\;\; B = \int \frac{u\,R}{uv' - u'v}\,dx$$